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CENTRAL ELEMENTS OF THE JENNINGS BASIS AND CERTAIN MORITA INVARIANTS

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ABSTRACT. From Morita theoretic viewpoint, computing Morita invariants is important. We proved that the intersection of the center and the nth socle $ZS^n(A) := Z(A) \cap \mathrm{Soc}^n(A)$ of a finite dimensional algebra A is Morita invariant; This is a generalization of important Morita invariants, the center $Z(A)$ and the Reynolds ideal $ZS^1(A)$.

As an example, we also studied $ZS^n(FP)$ for the group algebra FP of a finite p-group P over a field F of positive characteristic p. Such an algebra has a basis along the radical filtration, known as the Jennings basis. We show sufficient conditions under which an element of the Jennings basis is central and a lower bound for the dimension of $ZS^n(FP)$ for every positive integer n. Equalities hold for $1 \leq n \leq p$ if P is powerful. As a corollary we have $\mathrm{Soc}^p(FP) \subseteq Z(FP)$ if P is powerful.

This is a report of a talk based on [Sakurai, arXiv:1701.03799v2].

CONTENTS

1. INTRODUCTION

From Morita theoretic viewpoint, computing Morita invariants is important to distinguish algebras that are not Morita equivalent. We show that the

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intersection of the center and the nth right socle

$$ZS^n(A) := Z(A) \cap \mathrm{Soc}^n(A) \tag{1.1}$$

is Morita invariant for a finite dimensional algebra A (Theorem 2.2) in Section 2. This is a generalization of important Morita invariants, the center $Z(A)$ and the Reynolds ideal $ZS^1(A)$. The other way of generalization is known as the Külshammer ideals or the generalized Reynolds ideals for a symmetric algebra; For more details we refer the reader to the survey by Zimmermann [17].

For the group algebra FG of a finite group G over an algebraically closed field F of positive characteristic p, the dimension of the center $Z(FG)$ and the Reynolds ideal $ZS^1(FG)$, respectively, equal the number of irreducible ordinary characters $k(G)$ and the irreducible modular characters $\ell(G)$ (see [9, Lemmas 3 and 4]). (See also (4.3) for $n = 2$.) Moreover the conjugacy class sums form a basis for the center and the p-regular section sums form a basis for the Reynolds ideal (see [5, Satz D]; see also [9, Lemma 3] and [11, Theorem 1]). These are summarized in Table 1.1 where t denotes the Loewy length of FG.

TABLE 1.1. What is known about $ZS^n(FG)$.

	dimension (representation-theoretic)	basis (group-theoretic)
$ZS^t(FG)$	$k(G)$	conjugacy class sums
$ZS^n(FG)$	unknown	unknown
$ZS^2(FG)$	$\ell(G) + \sum_{S:\ \mathrm{simple}} \dim \mathrm{Ext}^1(S,S)$	unknown*
$ZS^1(FG)$	$\ell(G)$	p-regular section sums

As $ZS^n(FG)$ is a generalization of such, we want to know what is the dimension and what a basis can be. One of manageable examples to compute socle series is the group algebra FP of a finite p-group P over a field F of positive characteristic p. Jennings constructed a basis of FP along the radical filtration (Theorem 3.5) and it follows that radical series coincides with socle series (Theorem 3.6). Such basis is known as the Jennings basis (Definition 3.7). To give a lower bound for the dimension of $ZS^n(FP)$ we ask a question: When is an element of the Jennings basis central? In Section 4 we give sufficient conditions under which an element of the Jennings basis is central and a lower bound for the dimension of $ZS^n(FP)$ for every positive integer n (Theorem 4.1). This lower bound is sharp; We also proved that the equalities hold for $1 \leq n \leq p$ if P is powerful (Theorem 4.4). As a corollary we have $\mathrm{Soc}^p(FP) \subseteq Z(FP)$ if P is powerful (Corollary 4.5). Examples in Section 5 show that the corollary is best possible.

2. MORITA INVARIANTS

Inspired by the proof of Morita invariance of the Külshammer ideals by Héthelyi et al. [2, Proposition 5.1] (see also [16, Theorem 1] for their derived

*Except finite p-groups; see Remark 4.7.

invariance), we showed that the intersection of the center and the nth right socle $ZS^n(A) = Z(A) \cap \mathrm{Soc}^n(A)$, an ideal of the center, is a Morita invariant for a finite dimensional algebra A over a field. Note that this is left-right symmetric unlike socle. In the following $\mathrm{Rad}^n(A)$ denotes the nth Jacobson radical of A.

Lemma 2.1. *Let A be a finite dimensional algebra over a field and suppose $e \in A$ is a full idempotent. Then $\mathrm{Rad}^n(eAe) = \mathrm{Rad}^n(A) \cap eAe$ for every positive integer n.*

Theorem 2.2 (Sakurai [12]). *Let A and B be Morita equivalent finite dimensional algebras over a field. Then there is an algebra isomorphism $Z(A) \to Z(B)$ such that the following diagram commutes for all positive integer n. In particular, $ZS^n(A)$ are Morita invariants.*

$$\begin{array}{ccc} Z(A) & \longrightarrow & Z(B) \\ \uparrow & & \uparrow \\ ZS^n(A) & \longrightarrow & ZS^n(B) \end{array}$$

3. Jennings theory

As we established Morita invariance of $ZS^n(A)$ in Theorem 2.2, we want to determine the invariants for special cases. We hereafter study a group algebra FP of a finite p-group P over a field F of positive characteristic p. In this section we collect results of the Jennings theory.

Definition 3.1. For a positive integer i we define the ith *dimension subgroup* (or *Jennings subgroup*) of P by

$$D_i := \{ u \in P \mid u - 1 \in \mathrm{Rad}^i(FP) \}.$$

Remark 3.2. Although the dimension subgroups are defined ring-theoretically, those can be computed group-theoretically by Theorem 3.6(iv).

Lemma 3.3.

(i) *Every dimension subgroup is a characteristic subgroup.*
(ii) *Every successive quotient of the dimension subgroups is an elementary abelian p-group.*

Notation 3.4. Let D_i be the dimension subgroups of P. For a successive quotient of the dimension subgroups D_i/D_{i+1} of p-rank r_i, we fix elements $u_{i1}, \dots, u_{ir_i} \in D_i$ such that

$$D_i/D_{i+1} = \langle u_{i1}D_{i+1} \rangle \times \cdots \times \langle u_{ir_i}D_{i+1} \rangle.$$

Set $\ell := \min\{ i \geq 1 \mid D_i = 1 \}$, $\Lambda := \{ (i,j) \mid 1 \leq i < \ell,\ 1 \leq j \leq r_i \}$, and $M := \{0, 1, \dots, p-1\}^{\Lambda}$. For $\mu = (\mu_{ij}) \in M$ define

$$w(\mu) := \sum_{(i,j)\in\Lambda} i\mu_{ij} \quad \text{and} \quad z^{\mu} := \prod_{(i,j)\in\Lambda}{}' z_{ij}^{\mu_{ij}}, \tag{3.1}$$

where $z_{ij} := u_{ij} - 1$ and the product $\prod'$ is taken in lexicographic order. For an integer k define

(3.2) $$M_k := \{ \mu \in M \mid i \geq k \implies \mu_{ij} = p-1 \text{ for all } (i,j) \in \Lambda \}$$

and $\mu_k \in M_k$ by

(3.3) $$(\mu_k)_{ij} = \begin{cases} p-1 & (i \geq k) \\ 0 & (i < k). \end{cases}$$

Theorem 3.5 (Jennings [3]). *For every non-negative integer n we have*

$$\operatorname{Rad}^n(FP) = \bigoplus_{\substack{\mu \in M \\ w(\mu) \geq n}} Fz^{\mu}.$$

Theorem 3.6 (Jennings [3]).

(i) *The Loewy length $\ell\ell(FP)$ of FP equals $1 + w(\mu_1)$.*

(ii) *FP is rigid:* $\operatorname{Soc}^n(FP) = \operatorname{Rad}^{\ell\ell(FP)-n}(FP)$ *for every* $0 \leq n \leq \ell\ell(FP)$.

(iii) $\operatorname{Soc}^n(FP) = \bigoplus_{\substack{\mu \in M \\ w(\mu_1) - w(\mu) < n}} Fz^{\mu}$ *for every* $n \geq 0$.

(iv) $D_1 = P$ *and* $D_i = (D_{\lceil i/p \rceil})^p [D_{i-1}, P]$ *for every* $i > 1$.

Definition 3.7. The basis $\{ z^{\mu} \mid \mu \in M \}$ of FP is said to be the *Jennings basis.*

4. Main theorems

As promised, we give sufficient conditions under which an element of the Jennings basis is central and a lower bound for the dimension of the Morita invariant $ZS^n(FP) = Z(FP) \cap \operatorname{Soc}^n(FP)$ for every positive integer n in Theorem 4.1.

Theorem 4.1 (Sakurai [12]). *Let F be a field of positive characteristic p and P a finite p-group. Suppose k is a positive integer that satisfies $D_k \geq [P, P]$ where D_k denotes the kth dimension subgroup of P (recall Definition 3.1). Then, with Notation 3.4 and (1.1), we have*

(4.1) $$ZS^{n_k}(FP) \supseteq \bigoplus_{\mu \in M_k} Fz^{\mu}$$

where $n_k := 1 + w(\mu_1) - w(\mu_k)$. In particular, for every positive integer n we have

(4.2) $$ZS^n(FP) \supseteq \bigoplus_{\substack{\mu \in M_k \\ w(\mu_1) - w(\mu) < n}} Fz^{\mu}.$$

Remark 4.2. Note that such k always exists: $D_2 \geq [P, P]$. Note also that the dimension of the right hand side of (4.1) equals $|M_k| = |P/D_k|$.

See Section 5 for concrete examples. We can show that those coincide under the following conditions.

Definition 4.3. A finite p-group P is said to be *powerful* if $[P,P] \leq P^p$ and $p > 2$, or $[P,P] \leq P^4$ and $p = 2$.

Theorem 4.4 (Sakurai [12]). *If P is powerful then for every $1 \leq n \leq p$ we have*

$$ZS^n(FP) = \bigoplus_{\substack{\mu \in M_2 \\ w(\mu_1)-w(\mu)<n}} Fz^{\mu}.$$

Corollary 4.5 (Sakurai [12]). *If P is powerful then we have*

$$\mathrm{Soc}^p(FP) \subseteq Z(FP).$$

We give some remarks concerning these theorems in the rest of this section.

Remark 4.6. It is known that $\mathrm{Soc}^2(A) \subseteq Z(A)$ for a finite dimensional split-local symmetric algebra A. This can be traced back to Müller [7, Proof of Lemma 2]. (For a simple proof see, for example, [1, Lemma 2.2].)

Remark 4.7. Note that $ZS^n(FP)$ can be written down explicitly for $n = 1, 2$:

$$ZS^1(FP) = Fz^{\mu_1}$$
$$ZS^2(FP) = Fz^{\mu_1} \oplus \bigoplus_{1 \leq j \leq r_1} Fz^{\mu_1 - \delta_j}$$

where $\delta_j \in M$ is an element with 1 at $(1, j)$ and 0 otherwise. This is due to the Jennings theory and Remark 4.6.

Remark 4.8. Huppert raised a question that are the dimensions of Loewy layers of FP unimodal? Negative answer is given by Manz-Staszewski [6] and Stammbach-Stricker [15], independently. Nevertheless positive answer is known under certain condition; Shalev proved that the dimensions of Loewy layers are unimodal for powerful p-group if $p > 2$ (see [13, Proposition 4.1]). Hence it is reasonable to assume powerful as the Loewy series is well-behaved.

Remark 4.9. Let A be a block of a finite group algebra (or a finite dimensional symmetric algebra) over an algebraically closed field. Okuyama obtained the dimension of $ZS^2(A)$ that

$$\dim ZS^2(A) = \dim ZS^1(A) + \sum_S \dim \mathrm{Ext}_A^1(S, S) \tag{4.3}$$

where the sum is taken over a set of representatives of isoclasses of simple A-modules [8]. (See also [4, Theorem 2.1] which is written in English.)

Recently Otokita obtained an upper bound for the dimension of $ZS^n(A)$ that

$$\dim ZS^n(A) \leq \sum_S c(P_S/\operatorname{Rad}^n(P_S), S) \tag{4.4}$$

where the sum is taken over a set of representatives of isoclasses of simple A-modules and $c(P_S/\operatorname{Rad}^n(P_S), S)$ denote the composition multiplicity of a simple module S in the factor module $P_S/\operatorname{Rad}^n(P_S)$ of the projective cover P_S of S [10, Theorem 1.1].

5. Examples

In this section we illustrate our results by examples. In particular, the examples show that Corollary 4.5 is best possible. In the following we consider group algebras of extra-special p-groups of order p^3 for odd prime p over a field F of characteristic p.

5.1. **Extra-special p-group p_+^{1+2}.** Let P be an extra-special p-group of order p^3 and exponent p defined by

$$P := p_+^{1+2} = M(p) = \langle\, a, b, c \mid a^p = b^p = c^p = [a,c] = [b,c] = 1, [b,a] = c\,\rangle$$

for odd prime p and set $x := a-1$, $y := b-1$, and $z := c-1$. Then we can show that

$$\operatorname{Rad}^n(FP) = \bigoplus_{\substack{0\leq i,j,k<p \\ i+j+2k\geq n}} Fx^iy^jz^k \tag{5.1}$$

$$Z(FP) = \bigoplus_{0\leq k<p-1} Fz^k \oplus \bigoplus_{0\leq i,j<p} Fx^iy^jz^{p-1}. \tag{5.2}$$

In particular, for $p=3$, we have

$$FP \sim \begin{bmatrix} & & & & \mathbf{1} \\ & & & x & y \\ & x^2 & xy & y^2 & \boldsymbol{z} \\ & x^2y & xy^2 & xz & yz \\ x^2y^2 & x^2z & xyz & y^2z & \boldsymbol{z^2} \\ & x^2yz & xy^2z & \boldsymbol{xz^2} & \boldsymbol{yz^2} \\ & x^2y^2z & \boldsymbol{x^2z^2} & \boldsymbol{xyz^2} & \boldsymbol{y^2z^2} \\ & & & \boldsymbol{x^2yz^2} & \boldsymbol{xy^2z^2} \\ & & & & \boldsymbol{x^2y^2z^2} \end{bmatrix} \tag{5.3}$$

which mean that ith row consists of the elements of the Jennings basis lying in $\operatorname{Rad}^{i-1}(FP) \setminus \operatorname{Rad}^i(FP)$ and bold letters show that the elements are central. Note that P is not powerful and $x^2y^2z \in \operatorname{Soc}^3(FP) \setminus Z(FP)$. Hence the assertion of Corollary 4.5 does not hold without the assumption that P is powerful.

TABLE 5.1. Relevant numbers of the main theorems for $P = 3^{1+2}_{+}$.

n	1	2	3	4	5	6	7	8	9
$\dim \mathrm{Soc}^n(FP)$	1	3	7	11	16	20	24	26	27
$\dim ZS^n(FP)$	1	3	6	8	9	9	10	10	11
$\#\{\mu \in M_2 \mid w(\mu_1) - w(\mu) < n\}$	1	3	6	8	9	9	9	9	9

5.2. **Extra-special p-group p^{1+2}_{-}.** Let P be an extra-special p-group of order p^3 and exponent p^2 defined by

$$P := p^{1+2}_{-} = M_3(p) = \langle\, a, b \mid a^p = b^{p^2} = 1, b^a = b^{1+p} \,\rangle$$

for odd prime p and set $x := a - 1$, $y := b - 1$, and $z := c - 1$ where $c = b^p$. Then we can show that

$$\mathrm{Rad}^n(FP) = \bigoplus_{\substack{0 \le i,j,k < p \\ i+j+pk \ge n}} Fx^i y^j z^k \tag{5.4}$$

$$Z(FP) = \bigoplus_{0 \le k < p-1} Fz^k \oplus \bigoplus_{0 \le i,j < p} Fx^i y^j z^{p-1}. \tag{5.5}$$

In particular, for $p = 3$, we have

$$FP \sim \left[\begin{array}{ccc} & & \mathbf{1} \\ & x & y \\ x^2 & xy & y^2 \\ x^2y & xy^2 & z \\ x^2y^2 & xz & yz \\ x^2z & xyz & y^2z \\ x^2yz & xy^2z & \boldsymbol{z^2} \\ x^2y^2z & \boldsymbol{xz^2} & \boldsymbol{yz^2} \\ \boldsymbol{x^2z^2} & \boldsymbol{xyz^2} & \boldsymbol{y^2z^2} \\ & \boldsymbol{x^2yz^2} & \boldsymbol{xy^2z^2} \\ & & \boldsymbol{x^2y^2z^2} \end{array}\right] \tag{5.6}$$

where the convention is the same as (5.3). Note that P is powerful and $x^2y^2z \in \mathrm{Soc}^{3+1}(FP) \setminus Z(FP)$. Hence even if P is powerful a stronger assertion of Corollary 4.5 that $\mathrm{Soc}^{p+1}(FP) \subseteq Z(FP)$ is false in general.

TABLE 5.2. Relevant numbers of the main theorems for $P = 3^{1+2}_{-}$.

n	1	2	3	4	5	6	7	8	9	10	11
$\dim \mathrm{Soc}^n(FP)$	1	3	6	9	12	15	18	21	24	26	27
$\dim ZS^n(FP)$	1	3	6	8	9	9	9	10	10	10	11
$\#\{\mu \in M_3 \mid w(\mu_1) - w(\mu) < n\}$	1	3	6	8	9	9	9	9	9	9	9

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